Equations: In Plane Polar Coordinates: $\hat{r} = \hat{x}cos\theta + \hat{y}sin\theta$ and $\hat{\theta} = -\hat{x}sin\theta + \hat{y}cos\theta$

1. (10) Let's get the ol' pencils moving:
   (a) (4) Compare the vector torque about a point to the angular momentum about a point.
   (b) (4) Compare the divergence of a vector point function to the curl of a vector point function.
   (c) (2) Does it make sense to take the gradient of a vector point function? (yes, no)

2. For the simple harmonic oscillator having the equation $m\ddot{x} + b\dot{x} + kx = 0$
   (a) (11) Define $m$, $b$, and $k$. Draw a free body diagram illustrating all forces. Where is $x$
   measured from? Use your free body diagram to derive the above equation.
   (b) (3) Define in terms of constants above: $\gamma$, $\omega_0$, and $\omega_1$.
   (c) (6) Write down the solution for $x$ and plot $x$ versus $t$ for the cases: (i) $\gamma < \omega_0$ (ii) $\gamma = \omega_0$ (iii) $\gamma > \omega_0$

3. A particle is “trapped” in the following potential energy well:

   (a) (10) Describe the motion (including points) for (i) $E = +2.0 \text{ J}$ (ii) $E = +1.0 \text{ J}$ (iii) $E = -0.5 \text{ J}$ and
   (iv) $E = -1.0 \text{ J}$
   (b) (4) If the mass of the particle is 100 grams, find its speed if $E = 0 \text{ J}$ and $x = 3.0 \text{ m}$.
   (c) (6) Draw the corresponding graph for the force.

4. (15) (a) Which of the following forces are conservative (i) $F = -kx$ (ii) $F = -\frac{GmM}{x^2}$ (iii) $F = -mg$
   (b) For those that are, find and plot the potential energy function which corresponds to it.

5. (10) In plane polar coordinates, (a) distinguish between the following: (i) $r$ (ii) $\hat{r}$ and (iii) $\hat{\theta}$.
   (b) Use a diagram to illustrate how $r$ and $\theta$ relate mathematically to $x$ and $y$.
   (c) Derive $\frac{d\hat{r}}{d\theta} = \hat{\theta}$ and $\frac{d\hat{\theta}}{d\theta} = -\hat{r}$

6. Assume you have a critically damped simple harmonic oscillator that at $t = 0$, starts at $x = x_0$, has
   an initial velocity of $v_0$, and is driven by an external force $F = F_0 e^{-at}$ where $F_0$ and $a$ are constants.
   (a) (5) Write down the homogeneous solution. (b) (15) Find the particular solution. (c) (5)
   Explain how you would find the motion of the oscillator $x(t)$, and take it the first couple steps.
#1 (a) Vector Torque about point O
\[ \mathbf{T} = \mathbf{r} \times \mathbf{F} \]
\[ L = \mathbf{r} \times \mathbf{p} \]
Angular momentum about point O

(b) \[ \nabla \cdot \mathbf{A} = \lim_{r \to 0} \frac{\mathbf{A} \cdot d\mathbf{r}}{r^2} \] A Scalar!

\[ \mathbf{A} = \lim_{r \to 0} \frac{\text{Net Outward Flux through solid surface}}{\text{Volume } V \text{ enclosed}} \]

The component of the curl vector in a particular direction \( \mathbf{A} \) is
\[ \mathbf{A} \cdot \mathbf{n} = \lim_{\Delta S \to 0} \frac{\mathbf{A} \cdot d\mathbf{r}}{\Delta S} \]

(c) No!

#2 \[ m \ddot{x} + b \dot{x} + kx = 0 \]
(a) \( m \) is the mass
(b) \( b \) is the damping constant
(c) \( k \) is the spring constant

Equilibrium Position
\[ m \ddot{x} = 0 \]
\[ x = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \]

(b) \( \omega_0 = \sqrt{\frac{k}{m}} \)
\[ m \ddot{x} + b \dot{x} + kx = 0 \]
\[ s = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \]
\[ \omega = \omega_0 \sqrt{1 - \frac{\omega^2}{\omega_0^2}} \]
\[ x(t) = e^{-\frac{b}{2m}t} \left( x_0 \cos \left( \omega \sqrt{1 - \frac{\omega^2}{\omega_0^2}} t \right) + \frac{b}{2m} x_0 \sin \left( \omega \sqrt{1 - \frac{\omega^2}{\omega_0^2}} t \right) \right) \]

(c) \( x(t) = x_0 e^{-\frac{b}{2m}t} \cos \left( \omega \sqrt{1 - \frac{\omega^2}{\omega_0^2}} t \right) \]

#4 All forces are from F = \( kx \) and are conservative

(i) \[ -kx = \frac{dV}{dx} \]
\[ V = -\frac{1}{2} kx^2 \]

(ii) \[ \frac{dV}{dx} = -\frac{GMm}{x^2} \]
\[ V = \frac{GMm}{x} \]

(iii) \[ \frac{dV}{dx} = mg \]
\[ V = mgx \]

#5
\[ r = \sqrt{x^2 + y^2} \]
\[ \theta = \tan^{-1} \frac{y}{x} \]

(i) \( r \) is a scalar, the distance of point from origin
\[ r = \sqrt{x^2 + y^2} \]

(ii) \( r \) is a unit vector directed radially outward
\[ \mathbf{r} = r \mathbf{e}_r \]

(iii) \[ \mathbf{r} \times \mathbf{r} = \mathbf{0} \]
\[ \hat{r} \times \hat{r} = \mathbf{0} \]

Here \[ \frac{d\mathbf{r}}{d\theta} = \hat{x}(-\sin\theta) + \hat{y} \cos\theta \]
\[ \frac{d\hat{r}}{d\theta} = -\hat{x} \cos\theta + \hat{y} (-\sin\theta) \]
\[ \frac{d\theta}{d\theta} = -\hat{x} \cos\theta + \hat{y} (-\sin\theta) \]

Please see next page for #6
# 6 Critically DAMPED Oscillator! Driven by $F = F_0 e^{-at}$

Immediately (a) \( x_H = (C_1 + C_2 t) e^{-\delta t} \) where \( \delta = \frac{b}{2m} + 5 \)

(b) \( \text{Driven!} \quad m \ddot{x} + b \dot{x} + k x = F_0 e^{-at} \quad \text{AND Note} \quad \delta = \omega_0 \quad \text{since critically damped} \)

Assume \( x_p = Ae^{-at} \)

Plug in and solve for \( A \)

\[
\dot{x}_p = A(-\delta)e^{-at} \\
\ddot{x}_p = A(-\delta)^2 e^{-at} \\
+ mA^2 e^{-at} - bAe^{-at} + kAe^{-at} = F_0 e^{-at}
\]

\( \Rightarrow A \left( ma^2 - b\delta + k \right) = F_0 \quad \Rightarrow \quad A = \frac{F_0}{ma^2 - b\delta + k} \)

\[ x_p = \frac{F_0 \cdot e^{-at}}{ma^2 - b\delta + k} \]

\( \Rightarrow \) \( x_T = x_H + x_p \)

Apply BC: (1) When \( t = 0, \quad x_T = X_0 \quad x_0 = C_1 + \frac{F_0}{ma^2 - b\delta + k} \quad \Rightarrow \quad C_1 = X_0 - \frac{F_0}{ma^2 - b\delta + k} \)

(2) When \( t = 0, \quad \dot{x} = \dot{X}_0 \)

\( \dot{V} = \frac{dx_T}{dt} = C_2 e^{-\delta t} + (C_1 + C_2 t)(-\delta) e^{-\delta t} + \frac{F_0}{ma^2 - b\delta + k} (-\delta) e^{-\delta t} \)

At \( t = 0 \)

\( \dot{V}_0 = C_2 - \delta C_1 - \frac{F_0 \alpha}{ma^2 - b\delta + k} \)

\( \Rightarrow \quad \Delta C_2 = \delta C_1 + \frac{F_0 \alpha}{ma^2 - b\delta + k} + \dot{V}_0 \)

Hence,

\[ x_T = \left\{ \begin{array}{l} \frac{X_0 - F_0}{ma^2 - b\delta + k} + \left[ \dot{V}_0 + \delta \left( \frac{X_0 - F_0}{ma^2 - b\delta + k} \right) + \frac{F_0 \alpha}{ma^2 - b\delta + k} \right] t \right\} e^{-\delta t} + \frac{F_0 e^{-at}}{ma^2 - b\delta + k} \]