I. Summary & Interpretation of the General Central Force

A. We found a central force to be conservative and that the corresponding potential energy $V(r) = -\int F(r) \, dr$. We also found both from the basic definitions and from the equations of motion that both the total energy $E$ and the angular momentum $L$ are constants of the motion. These constants and the initial coordinates $r_0$ and $\theta_0$ can be used in determining $r(t)$ and $\theta(t)$ from

$$\int_{r_0}^{r} \frac{dr}{\left[E - V(r) - \frac{L^2}{2m r^2}\right]^{1/2}} = \sqrt{\frac{2}{m}} \, t$$  \hspace{1cm} \text{and} \hspace{1cm} \theta = \theta_0 + \int_{0}^{t} \frac{L}{m r^2} \, dt$$

In practice, solving the above integrals is difficult but we can simplify them and their interpretation if we reduce them to a form we are more familiar with.

The equations of motion are

$$m \ddot{r} - m r \dot{\theta}^2 = F(r)$$  \hspace{1cm} (4)

$$m r \ddot{\theta} + 2 m \dot{r} \dot{\theta} = 0 \Rightarrow \ddot{\theta} = \frac{L}{m r^2}$$  \hspace{1cm} (5)

Iterating (4)

$$m \ddot{r} - m r \dot{\theta}^2 = F(r)$$  \hspace{1cm} (6)

$$\dot{r}^2 \, m \ddot{r} = F(r) + \frac{L^2}{m r^3}$$  \hspace{1cm} (7)

From (1)

$$V_{\text{cent}} = -\int_{0}^{r} F(r) \, dr = -\int \frac{L^2}{m r^3} \, dr = -\frac{L^2}{m} \frac{1}{r^2} \bigg|_{0}^{\infty} = \frac{L^2}{2m r^2}$$

Let $V_{\text{eff}} = V(r) + V_{\text{cent}} = V(r) + \frac{L^2}{2m r^2}$

Hence

$$\int_{r_0}^{r} \frac{dr}{\left[E - V(r) - \frac{L^2}{2m r^2}\right]^{1/2}} = \int_{r_0}^{r} \frac{dr}{\left[E - V_{\text{eff}}\right]^{1/2}} = \sqrt{\frac{2}{m}} \, t$$  \hspace{1cm} (9)

Equation (9) is the same as we determined earlier for a 1-dimensional conservative force and the assumptions carry over!
B. All interpretations carried through earlier can now be made using \( V_{\text{eff}}(r) \). In particular all graphs on p 3-4 hold and we can have oscillations about a local minimum in \( V_{\text{eff}} \). Again \( k = \omega^2 m = \left. \frac{d^2 V_{\text{eff}}}{dr^2} \right|_{r_0} \)

\[
\omega^2 = \frac{1}{m} \left. \frac{d^2 V_{\text{eff}}}{dr^2} \right|_{r_0}
\]

C. For the inverse square (law) force:

\[
F = \frac{k}{r^2} \hat{r}
\]

\[
V(r) = -\int_{\infty}^{r} \frac{k}{r^2} dr = -k \left[ \frac{r^{-1}}{-1} \right]_{\infty}^{r} = \frac{k}{r}
\]

Hence \( V_{\text{eff}} = \frac{k}{r} + \frac{\ell^2}{2m r^2} \)

\[ \text{Orbital motion} \]